On the Stability Functional for Conservation Laws

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Abstract

This note is devoted to the explicit construction of a functional defined on all pairs of $\mathbf{L^1}$ functions with small total variation, which is equivalent to the $\mathbf{L^1}$ distance and non increasing along the trajectories of a given system of conservation laws. Two different constructions are provided, yielding an extension of the original stability functional by Bressan, Liu and Yang.

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1 Introduction

Let the smooth map $f:\Omega \mapsto \mathbb{R}^n$ define the strictly hyperbolic system of conservation laws

$$\partial_t u + \partial_x f(u) = 0 \tag{1.1}$$

where $t > 0$, $x \in \mathbb{R}$ and $u \in \Omega$, with $\Omega \subseteq \mathbb{R}^n$ being an open set.

Most functional theoretic methods fail to tackle these equations, essentially due to the appearance of shock waves. Since 1965, the Glimm functional [13] has been a major tool in any existence proof for (1.1) and related equations. More recently, an analogous role in the proofs of continuous dependence has been played by the stability functional Φ introduced in [7, 20, 21], see also [4]. The functional Φ has been widely used to prove the $\mathbf{L^1}$–Lipschitz dependence of solutions to (1.1) (and related problems) from initial data having small total variation, see for example [1, 2, 10, 11, 15, 16]. Special cases comprising data with large total variation are considered in [8, 14, 17, 18, 19]. Nevertheless, the use of Φ is hindered by the necessity of introducing specific approximate solutions, namely the ones based

either on Glimm scheme [13] or on the wave front tracking algorithm [4, 12]. The present paper makes the use of the stability functional Φ *independent* from any kind of approximate solutions. The present construction allows to simplify several parts of the cited papers, where the presentation of the stability functional needs to be preceded by the introduction of all the machinery related to Glimm's scheme or wave front tracking approximations, see for instance [9].

We extend the stability functional to all $\mathbf{L^1}$ functions with sufficiently small total variation. This construction is achieved in two different ways. First, we use general piecewise constant functions and a limiting procedure, without resorting to any sort of approximate solutions. Secondly, we exploit the *wave measures*, see [4, Chapter 10] and give an equivalent definition that does not require any limiting procedure. Furthermore, we prove its lower semicontinuity.

With reference to [4] for the basic definitions related to (1.1), we state the main result of this paper.

Theorem 1.1 *Let f generate a Standard Riemann Semigroup S on the domain $\mathcal{D}_\delta$ defined at* (2.6)*. Then, the functional Ξ defined at* (3.4) *by means of piecewise constant functions coincides with the one defined at* (4.5) *by means of wave measures. Moreover, it enjoys the following properties:*

(i) Ξ *is equivalent to the* $\mathbf{L^1}$ *distance, i.e. there exists a $C > 0$ such that for all $u, \tilde{u} \in \mathcal{D}_\delta$*

$$\frac{1}{C} \cdot \|u - \tilde{u}\|_{\mathbf{L^1}} \leq \Xi(u, \tilde{u}) \leq C \cdot \|u - \tilde{u}\|_{\mathbf{L^1}}.$$

(ii) Ξ *is non increasing along the semigroup trajectories, i.e. for all $u, \tilde{u} \in \mathcal{D}_\delta$ and for all $t \geq 0$*

$$\Xi(S_t u, S_t \tilde{u}) \leq \Xi(u, \tilde{u}).$$

(iii) Ξ *is lower semicontinuous with respect to the* $\mathbf{L^1}$ *norm.*

Taking advantage of the machinery presented below, we also extend the classical Glimm functionals [4, 13] to general $\mathbf{L^1}$ functions with small total variation and prove their lower semicontinuity, recovering some of the results in [3], but with a shorter proof.

2 Notation and Preliminary Results

Our reference for the basic definitions related to systems of conservation laws is [4]. We assume throughout that $0 \in \Omega$, with Ω open, and that

(F) $f \in \mathbf{C^4}(\Omega;\mathbb{R}^n)$, the system (1.1) is strictly hyperbolic with each characteristic field either genuinely nonlinear or linearly degenerate.

Let $\lambda_1(u),\ldots,\lambda_n(u)$ be the n real distinct eigenvalues of $Df(u)$, indexed so that $\lambda_j(u) < \lambda_{j+1}(u)$ for all j and u. The j-th right eigenvector, normalized as in (2.1)–(2.2), is $r_j(u)$. Let $\sigma \mapsto R_j(\sigma)(u)$, respectively $\sigma \mapsto S_j(\sigma)(u)$, be the rarefaction curve, respectively the shock curve, exiting u, so that

$$\frac{\partial R_j(\sigma)(u)}{\partial \sigma} = r_j\left(R_j(\sigma)(u)\right) . \tag{2.1}$$

If the j-th field is linearly degenerate, then the parameter σ above is the arc-length. In the genuinely nonlinear case, see [4, Definition 5.2], we choose σ so that

$$\frac{\partial \lambda_j}{\partial \sigma}\left(R_j(\sigma)(u)\right) = k_j \quad , \quad \frac{\partial \lambda_j}{\partial \sigma}\left(S_j(\sigma)(u)\right) = k_j \tag{2.2}$$

where $k_1,\ldots,k_n$ are arbitrary positive fixed numbers. In [4] the choice $k_j = 1$ for all $j = 1,\ldots,n$ was used, while in [2] another choice was made to cope with diagonal dominant sources. The choice (2.2) preserves the properties underlined in [4, Remark 5.4] so that the estimates in [4, Chapter 8] still hold. Introduce the j-Lax curve

$$\sigma \mapsto \psi_j(\sigma)(u) = \left\{ \begin{array}{lll} R_j(\sigma)(u) & \text{if} & \sigma \geq 0 \\ S_j(\sigma)(u) & \text{if} & \sigma < 0 \end{array} \right.$$

and for $\boldsymbol{\sigma} \equiv (\sigma_1,\ldots,\sigma_n)$, define the map

$$\mathbf{\Psi}(\boldsymbol{\sigma})(u^-) = \psi_n(\sigma_n) \circ \ldots \circ \psi_1(\sigma_1)(u^-) .$$

By [4, § 5.3], given any two states $u^-, u^+ \in \Omega$ sufficiently close to 0, there exists a map E such that

$$(\sigma_1,\ldots,\sigma_n) = E(u^-,u^+) \quad \text{if and only if} \quad u^+ = \mathbf{\Psi}(\boldsymbol{\sigma})(u^-) . \tag{2.3}$$

Similarly, let $\mathbf{q} \equiv (q_1,\ldots,q_n)$ and define the map $\mathbf{S}$ by

$$\mathbf{S}(\mathbf{q})(u^-) = S_n(q_n) \circ \ldots \circ S_1(q_1)(u^-) \tag{2.4}$$

as the gluing of the Rankine–Hugoniot curves. For any two states u^-, u^+ as above, there exists a unique $\mathbf{q}$ such that $u^+ = \mathbf{S}(\mathbf{q})(u^-)$.

Let u be piecewise constant with finitely many jumps and assume that $\mathrm{TV}(u)$ is sufficiently small. Call $\mathcal{I}(u)$ the finite set of points where u has a jump. Let $\sigma_{x,i}$ be the strength of the i-th wave in the solution of the Riemann problem for (1.1) with data $u(x-)$ and $u(x+)$, i.e. $(\sigma_{x,1},\ldots,\sigma_{x,n}) = E\left(u(x-),u(x+)\right)$. Obviously if $x \notin \mathcal{I}(u)$ then $\sigma_{x,i} = 0$, for all $i = 1,\ldots,n$. As usual, $\mathcal{A}(u)$ denotes the set of approaching waves in u:

$$\mathcal{A}(u) = \left\{ \begin{array}{c} \left((x,i),(y,j)\right) \in \left(\mathcal{I}(u) \times \{1,\ldots,n\}\right)^2 : \\ x < y \text{ and either } i > j \text{ or } i = j, \text{ the } i\text{-th field} \\ \text{is genuinely non linear, } \min\left\{\sigma_{x,i},\sigma_{y,j}\right\} < 0 \end{array} \right\} .$$

As in [13] or [4, formula (7.99)], the linear and the interaction potential are

$$\mathbf{V}(u) = \sum_{x \in I(u)} \sum_{i=1}^{n} \left|\sigma_{x,i}\right| \quad \text{and} \quad \mathbf{Q}(u) = \sum_{\left((x,i),(y,j)\right) \in \mathcal{A}(u)} \left|\sigma_{x,i}\sigma_{y,j}\right| .$$

Moreover, let

$$\mathbf{\Upsilon}(u) = \mathbf{V}(u) + C_0 \cdot \mathbf{Q}(u) \tag{2.5}$$

where $C_0 > 0$ is the constant appearing in the functional of the wave–front tracking algorithm, see [4, Proposition 7.1]. Recall that C_0 depends only on the flow f and on the upper bound of the total variation of initial data.

Remark 2.1 *For fixed $x_1 < \ldots < x_{N+1}$, the maps*

$$\begin{array}{rcl} (u_1, \ldots, u_N) & \mapsto & \mathbf{V}\left(\sum_{\alpha=1}^{N} u_\alpha \, \chi_{[x_\alpha, x_{\alpha+1}[}\right) \\ (u_1, \ldots, u_N) & \mapsto & \mathbf{Q}\left(\sum_{\alpha=1}^{N} u_\alpha \, \chi_{[x_\alpha, x_{\alpha+1}[}\right) \end{array}$$

are Lipschitz continuous. Moreover, the Lipschitz constant of the maps

$$u_{\bar\alpha} \mapsto \mathbf{V}\left(\sum_{\alpha=1}^{N} u_\alpha \, \chi_{[x_\alpha, x_{\alpha+1}[}\right) \qquad u_{\bar\alpha} \mapsto \mathbf{Q}\left(\sum_{\alpha=1}^{N} u_\alpha \, \chi_{[x_\alpha, x_{\alpha+1}[}\right)$$

is bounded uniformly in N, $\bar\alpha$ and u_α for $\alpha \neq \bar\alpha$.

Finally, for $\delta > 0$ sufficiently small, we define

$$\begin{array}{c} \mathcal{D}_\delta^* = \left\{ v \in \mathbf{L^1}(\mathbb{R}, \Omega) \colon v \text{ is piecewise constant and } \mathbf{\Upsilon}(v) < \delta \right\} \\ \mathcal{D}_\delta = \operatorname{cl} \mathcal{D}_\delta^* \end{array} \tag{2.6}$$

where the closure is in the strong $\mathbf{L^1}$–topology. Unless otherwise stated, we always consider the right continuous representatives of maps in $\mathcal{D}_\delta$ and $\mathcal{D}_\delta^*$.

For later use, for $u \in \mathcal{D}_\delta$ and $\eta > 0$, introduce the set

$$B_\eta(u) = \left\{ v \in \mathbf{L^1}(\mathbb{R}; \Omega) \colon v \in \mathcal{D}_\delta^* \text{ and } \|v - u\|_{\mathbf{L^1}} < \eta \right\} . \tag{2.7}$$

Note that, by the definition of $\mathcal{D}_\delta$, $B_\eta(u)$ is not empty and if $\eta_1 < \eta_2$, then $B_{\eta_1}(u) \subseteq B_{\eta_2}(u)$. Recall the following fundamental result, proved in [6]:

Theorem 2.2 *Let f satisfy **(F)**. Then, there exists a positive δ_o such that the equation* (1.1) *generates for all $\delta \in \left]0, \delta_o\right[$ a Standard Riemann Semigroup (SRS) $S \colon [0, +\infty[\times \mathcal{D}_\delta \mapsto \mathcal{D}_\delta$, which is Lipschitz in $\mathbf{L^1}$.*

We refer to [4, Chapters 7 and 8] for the proof of the above result as well as for the definition and further properties of the SRS.

3 The Piecewise Constant Functions Approach

Extend the Glimm functionals to all $u \in \mathcal{D}_\delta$ as follows:

$$\bar{\mathbf{Q}}(u) = \lim_{\eta \to 0+} \inf_{v \in B_\eta(u)} \mathbf{Q}(v) \quad \text{and} \quad \bar{\mathbf{\Upsilon}}(u) = \lim_{\eta \to 0+} \inf_{v \in B_\eta(u)} \mathbf{\Upsilon}(v) \,. \tag{3.1}$$

The maps $\eta \to \inf_{v \in B_\eta(v)} \mathbf{Q}(v)$ and $\eta \to \inf_{v \in B_\eta(v)} \mathbf{\Upsilon}(v)$ are non increasing. Thus the limits above exist and

$$\bar{\mathbf{Q}}(u) = \sup_{\eta > 0} \inf_{v \in B_\eta(u)} \mathbf{Q}(v) \quad \text{and} \quad \bar{\mathbf{\Upsilon}}(u) = \sup_{\eta > 0} \inf_{v \in B_\eta(u)} \mathbf{\Upsilon}(v) \,.$$

We prove in Proposition 3.4 below that $\bar{\mathbf{Q}}$, respectively $\bar{\mathbf{\Upsilon}}$, coincides with $\mathbf{Q}$, respectively $\mathbf{\Upsilon}$, when evaluated on piecewise constant functions. Moreover, we prove in Corollary 4.4 that $\bar{\mathbf{Q}}$ also coincides with the functional intended in [5, formula (1.15)] or [4, formula (10.10)]. Preliminarily, we exploit the formulation (3.1) to prove directly the lower semicontinuity of $\mathbf{Q}$ and $\mathbf{\Upsilon}$.

Proposition 3.1 *The functionals $\bar{\mathbf{Q}}$ and $\bar{\mathbf{\Upsilon}}$ are lower semicontinuous with respect to the $\mathbf{L^1}$ norm.*

Proof. We prove the lower semicontinuity of $\bar{\mathbf{\Upsilon}}$, the case of $\bar{\mathbf{Q}}$ is analogous.

Fix u in $\mathcal{D}_\delta$. Let u_ν be a sequence in $\mathcal{D}_\delta$ converging to u in $\mathbf{L^1}$. Define $\varepsilon_\nu = \|u_\nu - u\|_{\mathbf{L^1}} + 1/\nu$. Fix $v_\nu \in B_{\varepsilon_\nu}(u_\nu)$ so that

$$\mathbf{\Upsilon}(v_\nu) \leq \inf_{v \in B_{\varepsilon_\nu}(u_\nu)} \mathbf{\Upsilon}(v) + \varepsilon_\nu \leq \bar{\mathbf{\Upsilon}}(u_\nu) + \varepsilon_\nu \,.$$

From $\|v_\nu - u\|_{\mathbf{L^1}} \leq \|v_\nu - u_\nu\|_{\mathbf{L^1}} + \|u_\nu - u\|_{\mathbf{L^1}} < 2\varepsilon_\nu$ we deduce $v_\nu \in B_{2\varepsilon_\nu}(u)$ and the proof is completed with the following estimates:

$$\begin{array}{rcccl} \displaystyle\inf_{v \in B_{2\varepsilon_\nu}(u)} \mathbf{\Upsilon}(v) & \leq & \mathbf{\Upsilon}(v_\nu) & \leq & \bar{\mathbf{\Upsilon}}(u_\nu) + \varepsilon_\nu \,; \\ \bar{\mathbf{\Upsilon}}(u) & = & \displaystyle\lim_{\nu \to +\infty} \inf_{v \in B_{2\varepsilon_\nu}(u)} \mathbf{\Upsilon}(v) & \leq & \displaystyle\liminf_{\nu \to +\infty} \bar{\mathbf{\Upsilon}}(u_\nu) \,. \end{array}$$

□

The next proposition contains in essence the reason why the Glimm functionals $\mathbf{Q}$ and $\mathbf{\Upsilon}$ decrease. Compute them on a piecewise constant function u and *"remove"* one (or more) of the values attained by u, then the values of both $\mathbf{Q}$ and $\mathbf{\Upsilon}$ decrease.

Let $u = \sum_{\alpha \in I} u_\alpha \, \chi_{[x_\alpha, x_{\alpha+1}[}$ be a piecewise constant function, with $u_\alpha \in \Omega$, $x_1 < x_2 < \ldots < x_{N+1}$ and I be a finite set of integers. Then, we say that $u_1, u_2, \ldots, u_N$ is the *ordered sequence* of the values attained by u and we denote it by $(u_\alpha \colon \alpha \in I)$.

Proposition 3.2 *Let $u, \check{u} \in \mathcal{D}_\delta^*$. If the ordered sequence of the values attained by u is $(u_\alpha \colon \alpha \in I)$ and the ordered sequence of the values attained by $\check{u}$ is $(u_\alpha \colon \alpha \in J)$, with $J \subseteq I$, then $\mathbf{Q}(\check{u}) \leq \mathbf{Q}(u)$ and $\mathbf{\Upsilon}(\check{u}) \leq \mathbf{\Upsilon}(u)$.*

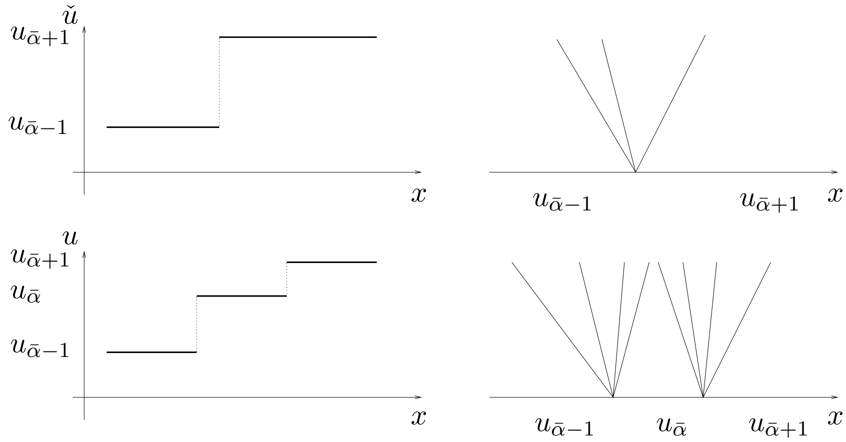

Figure 1: Proof of Proposition 3.2: $u_{\bar\alpha}$ is attained by u and not by $\check u$.

Proof. Consider the case $\sharp I = \sharp J + 1$, see also [4, Lemma 10.2, Step 1]. Then, the above inequalities follow from the usual Glimm interaction estimates [13], see Figure 1.

The general case follows recursively. □

The next lemma is a particular case of [4, Theorem 10.1]. However, the present construction allows to consider only the case of piecewise constant functions, allowing a much simpler proof.

Lemma 3.3 *The functionals* $\mathbf{Q}$ *and* $\boldsymbol{\Upsilon}$*, defined on* $\mathcal{D}^*_\delta$*, are lower semicontinuous with respect to the* $\mathbf{L^1}$ *norm.*

Proof. We consider only $\boldsymbol{\Upsilon}$, the case of $\mathbf{Q}$ being similar.

Let u_ν be a sequence in $\mathcal{D}^*_\delta$ converging in $\mathbf{L^1}$ to $u = \sum_\alpha u_\alpha \chi_{[x_\alpha, x_{\alpha+1}[} \in \mathcal{D}^*_\delta$ as $\nu \to +\infty$. By possibly passing to a subsequence, we may assume that $\boldsymbol{\Upsilon}(u_\nu)$ converges to $\liminf_{\nu\to+\infty} \boldsymbol{\Upsilon}(u_\nu)$ and that u_ν converges a.e. to u. Therefore, for all $\alpha = 1, \ldots, N$, we can select points $y_\alpha \in \,]x_\alpha, x_{\alpha+1}[$ so that $\lim_{\nu\to+\infty} u_\nu(y_\alpha) = u(y_\alpha) = u_\alpha$. Define

$$\check u_\nu = \sum_\alpha u_\nu(y_\alpha)\, \chi_{[x_\alpha, x_{\alpha+1}[} \,.$$

By Proposition 3.2, $\boldsymbol{\Upsilon}(\check u_\nu) \le \boldsymbol{\Upsilon}(u_\nu)$. The convergence $u_\nu(y_\alpha) \to u_\alpha$ for all α and Remark 2.1 allow to complete the proof. □

Proposition 3.4 *Let* $u \in \mathcal{D}^*_\delta$*. Then* $\bar{\mathbf{Q}}(u) = \mathbf{Q}(u)$ *and* $\bar{\boldsymbol{\Upsilon}}(u) = \boldsymbol{\Upsilon}(u)$*.*

Proof. We consider only $\boldsymbol{\Upsilon}$, the case of $\mathbf{Q}$ being similar.

Since $u \in \mathcal{D}^*_\delta$, we have that $u \in B_\eta(u)$ for all $\eta > 0$ and $\bar{\boldsymbol{\Upsilon}}(u) \le \boldsymbol{\Upsilon}(u)$. To prove the other inequality, recall that by the definition (3.1) of

$\bar{\mathbf{\Upsilon}}$, there exists a sequence v_ν of piecewise constant functions in $\mathcal{D}_\delta^*$ such that $v_\nu \to u$ in $\mathbf{L^1}$ and $\mathbf{\Upsilon}(v_\nu) \to \bar{\mathbf{\Upsilon}}(u)$ as $\nu \to +\infty$. By Lemma 3.3, $\mathbf{\Upsilon}(u) \leq \liminf_{\nu\to+\infty} \mathbf{\Upsilon}(v_\nu) \leq \bar{\mathbf{\Upsilon}}(u)$, completing the proof. □

Therefore, in the sequel we write $\mathbf{Q}$ for $\bar{\mathbf{Q}}$ and $\mathbf{\Upsilon}$ for $\bar{\mathbf{\Upsilon}}$.

Since we will need the explicit dependence of the Stability Functional on the various quantity it is made of, we introduce the following notations. If $\delta \in]0, \delta_o[$, for any $\bar{v} \in \mathcal{D}_\delta^*$, denote by $\bar{\sigma}_{x,i}$ the size of the i–wave in the solution of the Riemann Problem with data $\bar{v}(x-)$ and $\bar{v}(x+)$. Then define

$$A_j^-[\bar{v}](x) = \sum_{y\leq x} \left|\bar{\sigma}_{y,j}\right|, \quad A_j^+[\bar{v}](x) = \sum_{y>x} \left|\bar{\sigma}_{y,j}\right|, \qquad \text{for } j = 1, \ldots, n\,.$$

If the i-th characteristic field is linearly degenerate, then define $\mathbf{A}_i$ as

$$\mathbf{A}_i\left[\bar{v}\right](q,x) = \sum_{1\leq j<i} A_j^+\left[\bar{v}\right](x) + \sum_{i<j\leq n} A_j^-\left[\bar{v}\right](x).$$

While if the i-th characteristic field is genuinely nonlinear

$$\begin{aligned}\mathbf{A}_i[\bar{v}](q,x) \quad &= \quad \sum_{1\leq j<i} A_j^+[\bar{v}](x) + \sum_{i<j\leq n} A_j^-[\bar{v}](x) \\ &\quad\quad +A_i^+[\bar{v}](x)\cdot \chi_{[0,+\infty[}(q) + A_i^-[\bar{v}](x)\cdot\chi_{]-\infty,0[}(q)\end{aligned}$$

Now choose $v, \tilde{v}$ piecewise constant in $\mathcal{D}_\delta^*$ and define the weights

$$\begin{aligned}\mathbf{W}_i[v,\tilde{v}](q,x) \quad &= \quad 1 + \kappa_1\mathbf{A}_i[v](q,x) + \kappa_1\mathbf{A}_i[\tilde{v}](-q,x) \\ &\quad\quad +\kappa_1\kappa_2\left(\mathbf{Q}(v) + \mathbf{Q}(\tilde{v})\right)\,.\end{aligned} \tag{3.2}$$

the constants κ_1 and κ_2 being defined in [4, Chapter 8]. We may now define a slightly modified version of the stability functional, see [7, 20, 21] and also [4, Section 8.1]. Namely, we give a similar functional defined on all piecewise constant functions and without any reference to both ε–approximate front tracking solutions and non physical waves.

Define implicitly the function $\mathbf{q}(x) \equiv \left(q_1(x), \ldots, q_n(x)\right)$ by

$$\tilde{v}(x) = \mathbf{S}\left(\mathbf{q}(x)\right)\left(v(x)\right)$$

with $\mathbf{S}$ as in (2.4). The stability functional $\mathbf{\Phi}$ is

$$\mathbf{\Phi}(v,\tilde{v}) = \sum_{i=1}^n \int_{-\infty}^{+\infty} \left|q_i(x)\right| \cdot \mathbf{W}_i[v,\tilde{v}]\left(q_i(x), x\right)\, dx. \tag{3.3}$$

We stress that $\mathbf{\Phi}$ is slightly different from the functional Φ defined in [4, formula (8.6)]. Indeed, here *all* jumps in v or in $\tilde{v}$ are considered. There, on the contrary, exploiting the structure of ε-approximate front tracking

solutions, see [4, Definition 7.1], in the definition of Φ the jumps due to non physical waves are neglected when defining the weights A_i and are considered as belonging to a fictitious $(n+1)$-th family in the definition [4, formula (7.54)] of Q. To stress this difference, in the sequel we denote by Φ^ε the stability functional as presented in [4, Chapter 8].

Remark 3.5 *For fixed $x_1 < \ldots < x_{N+1}$, $\tilde{x}_1 < \ldots < \tilde{x}_{\tilde{N}+1}$, the map*

$$\begin{pmatrix} u_1, \ldots, u_N \\ \tilde{u}_1, \ldots, \tilde{u}_{\tilde{N}} \end{pmatrix} \mapsto \mathbf{\Phi}\left(\sum_{\alpha=1}^{N} u_\alpha\, \chi_{[x_\alpha, x_{\alpha+1}[}, \sum_{\alpha=1}^{\tilde{N}} \tilde{u}_\alpha\, \chi_{[\tilde{x}_\alpha, \tilde{x}_{\alpha+1}[}\right)$$

is continuous. Indeed, since both maps $q \mapsto q\chi_{[0,+\infty[}(q)$ and $q \mapsto q\chi_{]-\infty,0[}(q)$ are Lipschitz, for any fixed $x \in \mathbb{R}$ the integrand in (3.3) *depends continuously on $\{u_\alpha\}_{\alpha=1}^{N}$, $\{\tilde{u}_\alpha\}_{\alpha=1}^{\tilde{N}}$ and the Dominated Convergence Theorem applies.*

We now move towards the extension of $\mathbf{\Phi}$ to $\mathcal{D}_\delta$. Define

$$\mathbf{\Xi}_\eta(u, \tilde{u}) = \inf\left\{\mathbf{\Phi}(v, \tilde{v}) \colon v \in B_\eta(u) \text{ and } \tilde{v} \in B_\eta(\tilde{u})\right\}$$

The map $\eta \to \mathbf{\Xi}_\eta(u, \tilde{u})$ is non increasing. Thus, we may finally define

$$\mathbf{\Xi}(u, \tilde{u}) = \lim_{\eta \to 0+} \mathbf{\Xi}_\eta(u, \tilde{u}) = \sup_{\eta > 0} \mathbf{\Xi}_\eta(u, \tilde{u}) \,. \tag{3.4}$$

We are now ready to state the main result of this section.

Theorem 3.6 *Let f satisfy **(F)**. The functional $\mathbf{\Xi} \colon \mathcal{D}_\delta \times \mathcal{D}_\delta \mapsto [0, +\infty[$ defined in* (3.4) *enjoys the properties (i), (ii) and (iii) in Theorem 1.1.*

Here and in what follows, we denote by C positive constants dependent only on f and δ_0. We split the proof of the above theorem in several steps.

Lemma 3.7 *For all $u, \tilde{u} \in \mathcal{D}_\delta^*$, one has $\mathbf{\Xi}(u, \tilde{u}) \leq \mathbf{\Phi}(u, \tilde{u})$.*

Proof. By the definition (2.7) we have $u \in B_\eta(u)$ and $\tilde{u} \in B_\eta(\tilde{u})$ for all $\eta > 0$, hence $\mathbf{\Xi}_\eta(u, \tilde{u}) \leq \mathbf{\Phi}(u, \tilde{u})$ for all positive η. The lemma is proved passing to the limit $\eta \to 0+$. $\square$

Lemma 3.8 *Let u, $\check{u} \in \mathcal{D}_\delta^*$ and $q \in \mathbb{R}$. Assume that $\check{u}$ is given by*

$$\check{u} = \sum_{\alpha=1}^{N} u(y_\alpha) \chi_{[x_\alpha, x_{\alpha+1}[}$$

where $x_1 < \ldots < x_{N+1}$ and $y_\alpha \in [x_\alpha, x_{\alpha+1}[$ are given real points. Then,

$$\mathbf{A}_i\left[\check{u}\right](q, x) + \kappa_2 \mathbf{Q}(\check{u}) \leq \mathbf{A}_i\left[u\right](q, x) + \kappa_2 \mathbf{Q}(u) + C \cdot \left\|\check{u}(x) - u(x)\right\| .$$

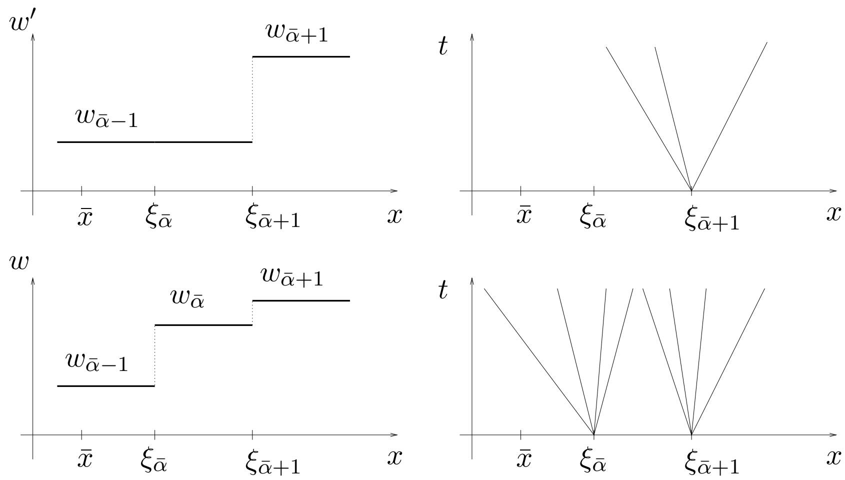

Figure 2: Exemplification of point 2 in the proof of Lemma 3.8.

Proof. Fix $\bar x \in \mathbb{R}$ and prove the above inequality passing from u to $\check u$ recursively applying three elementary operations:

1. w' is obtained from w only shifting the position of the points of jump but without letting any point of jump cross $\bar x$. More formally, if $w = \sum_\alpha w_\alpha \chi_{[\xi_\alpha,\xi_{\alpha+1}[}$ with $\xi_\alpha < \xi_{\alpha+1}$, $w' = \sum_\alpha w_\alpha \chi_{[\xi'_\alpha,\xi'_{\alpha+1}[}$ with $\xi'_\alpha < \xi'_{\alpha+1}$ and moreover $\bar x \in [\xi_{\bar\alpha},\xi_{\bar\alpha+1}[\cap \left[\xi'_{\bar\alpha},\xi'_{\bar\alpha+1}\right[$ for a suitable $\bar\alpha$, then

$$\mathbf{A}_i\left[w'\right](q,\bar x) + \kappa_2 \mathbf{Q}(w') = \mathbf{A}_i\left[w\right](q,\bar x) + \kappa_2 \mathbf{Q}(w)$$

Indeed, if all the jumps stay unchanged and no shock crosses $\bar x$, then nothing changes in the definition of $\mathbf{A}_i$ and $\mathbf{Q}$.

2. w' is obtained from w removing a value attained by w on an interval not containing $\bar x$, see Figure 2. More formally, if $w = \sum_\alpha w_\alpha \chi_{[\xi_\alpha,\xi_{\alpha+1}[}$ with $\xi_\alpha < \xi_{\alpha+1}$ and $\bar x \notin \ [\xi_{\bar\alpha},\xi_{\bar\alpha+1}[$, then

$$w' = \sum_{\alpha\neq\bar\alpha} w_\alpha\,\chi_{[\xi_\alpha,\xi_{\alpha+1}[} + w_{\bar\alpha-1}\,\chi_{[\xi_{\bar\alpha},\xi_{\bar\alpha+1}[}$$

or

$$w' = \sum_{\alpha\neq\bar\alpha} w_\alpha\,\chi_{[\xi_\alpha,\xi_{\alpha+1}[} + w_{\bar\alpha+1}\,\chi_{[\xi_{\bar\alpha},\xi_{\bar\alpha+1}[}\,.$$

In both cases,

$$\mathbf{A}_i\left[w'\right](q,\bar x) + \kappa_2 \mathbf{Q}(w') \leq \mathbf{A}_i[w](q,\bar x) + \kappa_2 \mathbf{Q}(w)\,.$$

Indeed, consider for example the situation in Figure 2. The two jumps at the points $\xi_{\bar\alpha}$ and $\xi_{\bar\alpha+1}$ in w are substituted by a single jump in w' at the point $\xi_{\bar\alpha+1}$. The points $\xi_{\bar\alpha}$ and $\xi_{\bar\alpha+1}$ are both to the right of $\bar x$, therefore the waves in w' at the point $\xi_{\bar\alpha+1}$ which appear in $\mathbf{A}_i[w'](q,\bar x)$ are of the

same families of the waves in w at the points $\xi_{\bar\alpha}$ and $\xi_{\bar\alpha+1}$ which appear in $\mathbf{A}_i[w](q,\bar x)$. Since all the other waves in $\mathbf{A}_i$ are left unchanged we have

$$\mathbf{A}_i[w'](q,\bar x) - \mathbf{A}_i[w](q,\bar x) \leq \sum_{j=1}^{n} \left|\sigma'_{\xi_{\bar\alpha+1},j} - \sigma_{\xi_{\bar\alpha},j} - \sigma_{\xi_{\bar\alpha+1},j}\right|$$

where

$$\begin{array}{rclcl} \sigma'_{\xi_{\bar\alpha+1},j} & = & E_j\left(w'\left(\xi_{\bar\alpha+1}-\right), w'\left(\xi_{\bar\alpha+1}+\right)\right) & = & E_j\left(w_{\bar\alpha-1}, w_{\bar\alpha+1}\right) \\ \sigma_{\xi_{\bar\alpha+1},j} & = & E_j\left(w\left(\xi_{\bar\alpha+1}-\right), w\left(\xi_{\bar\alpha+1}+\right)\right) & = & E_j\left(w_{\bar\alpha}, w_{\bar\alpha+1}\right) \\ \sigma_{\xi_{\bar\alpha},j} & = & E_j\left(w\left(\xi_{\bar\alpha}-\right), w\left(\xi_{\bar\alpha}+\right)\right) & = & E_j\left(w_{\bar\alpha-1}, w_{\bar\alpha}\right). \end{array}$$

Therefore, the increase in $\mathbf{A}_i$ evaluated at $\bar x$ is bounded by the interaction potential between the waves at $\xi_{\bar\alpha}$ and those at $\xi_{\bar\alpha+1}$ and is compensated by the decrease in $\kappa_2\mathbf{Q}$, as in the standard Glimm interaction estimates.

3. w' is obtained from w changing the value assumed by w in the interval containing $\bar x$. More formally, if $w = \sum_\alpha w_\alpha \chi_{[\xi_\alpha,\xi_{\alpha+1}[}$ with $\xi_\alpha < \xi_{\alpha+1}$ and $\bar x \in [\xi_{\bar\alpha}, \xi_{\bar\alpha+1}[$, then

$$w' = \sum_{\alpha\neq\bar\alpha} w_\alpha \chi_{[\xi_\alpha,\xi_{\alpha+1}[} + w'_{\bar\alpha}\chi_{[\xi_{\bar\alpha},\xi_{\bar\alpha+1}[}\,.$$

In this case

$$\begin{array}{rcl} \mathbf{A}_i\left[w'\right](q,\bar x) + \kappa_2\mathbf{Q}(w') & \leq & \mathbf{A}_i[w](q,\bar x) + \kappa_2\mathbf{Q}(w) + C\cdot\left\|w_{\bar\alpha} - w'_{\bar\alpha}\right\| \\ & \leq & \mathbf{A}_i[w](q,\bar x) + \kappa_2\mathbf{Q}(w) + C\cdot\left\|w(\bar x) - w'(\bar x)\right\|. \end{array}$$

This inequality follows from the Lipschitz dependence of $\mathbf{A}_i[w](q,\bar x)(\bar x)$ and $\mathbf{Q}(w)$ on $w_{\bar\alpha}$ with a Lipschitz constant independent from the number of jumps, see Remark 2.1.

Now for $\bar x \in [x_{\bar\alpha}, x_{\bar\alpha+1}[$ we can pass from u to the function $\bar w$ defined by

$$\bar w = \sum_{\alpha\neq\bar\alpha} u(y_\alpha)\chi_{[x_\alpha,x_{\alpha+1}[} + u(\bar x)\chi_{[x_{\bar\alpha},x_{\bar\alpha+1}[}$$

applying the first two steps a certain number of times. We obtain

$$\mathbf{A}_i[\bar w](q,\bar x) + \kappa_2\mathbf{Q}(\bar w) \leq \mathbf{A}_i[u](q,\bar x) + \kappa_2\mathbf{Q}(u)\,.$$

Finally with the third step we go from $\bar w$ to $\check u$ obtaining the estimate:

$$\begin{array}{rcl} \mathbf{A}_i[\check u](q,\bar x) + \kappa_2\mathbf{Q}(\check u) & \leq & \mathbf{A}_i[\bar w](q,\bar x) + \kappa_2\mathbf{Q}(\bar w) + C\cdot\left\|\check u(\bar x) - \bar w(\bar x)\right\| \\ & \leq & \mathbf{A}_i[u](q,\bar x) + \kappa_2\mathbf{Q}(u) + C\cdot\left\|\check u(\bar x) - u(\bar x)\right\| \end{array}$$

which proves the lemma. $\square$

Lemma 3.9 *Let $u,\ \check u,\ \tilde u,\ \check{\tilde u} \in \mathcal{D}^*_\delta$. Assume that $\check u$ and $\check{\tilde u}$ are given by*

$$\check u = \sum_{\alpha=1}^{N} u(y_\alpha)\chi_{[x_\alpha,x_{\alpha+1}[}, \qquad \check{\tilde u} = \sum_{\alpha=1}^{\tilde N} \tilde u(\tilde y_\alpha)\chi_{[\tilde x_\alpha,\tilde x_{\alpha+1}[}$$

where $x_1 < \ldots < x_{N+1}$, $y_\alpha \in [x_\alpha, x_{\alpha+1}[$, $\tilde x_1 < \ldots < \tilde x_{\tilde N+1}$, $\tilde y_\alpha \in [\tilde x_\alpha, \tilde x_{\alpha+1}[$ are fixed real points. Then,

$$\mathbf{\Phi}\left(\check u, \check{\tilde u}\right) \le \mathbf{\Phi}\left(u, \tilde u\right) + C\left(\left\|\check u - u\right\|_{\mathbf{L^1}} + \left\|\check{\tilde u} - \tilde u\right\|_{\mathbf{L^1}}\right).$$

Proof. Introduce $\mathbf{q}(x) = \big(q_1(x), \ldots, q_n(x)\big)$ and $\check{\mathbf{q}}(x) = \big(\check q_1(x), \ldots, \check q_n(x)\big)$ by $\tilde u(x) = \mathbf{S}\left(\mathbf{q}(x)\right)\big(u(x)\big)$ and $\check{\tilde u}(x) = \mathbf{S}\left(\check{\mathbf{q}}(x)\right)\big(\check u(x)\big)$ with $\mathbf{S}$ defined in (2.4).

$$\begin{aligned}
&\mathbf{\Phi}\left(\check u, \check{\tilde u}\right) - \mathbf{\Phi}\left(u, \tilde u\right) \\
= &\sum_{i=1}^{n}\int_{-\infty}^{+\infty}\left\{\left|\check q_i(x)\right|\mathbf{W}_i[\check u, \check{\tilde u}]\left(\check q_i(x), x\right) - \left|q_i(x)\right|\mathbf{W}_i[\check u, \check{\tilde u}]\left(q_i(x), x\right)\right\} dx \\
&+ \sum_{i=1}^{n}\int_{-\infty}^{+\infty}\left|q_i(x)\right|\left\{\mathbf{W}_i[\check u, \check{\tilde u}]\left(q_i(x), x\right) - \mathbf{W}_i[u, \tilde u]\left(q_i(x), x\right)\right\} dx\,.
\end{aligned}$$

Since the map $q \mapsto q \cdot \mathbf{W}_i[\check u, \check{\tilde u}](q, x)$ is uniformly Lipschitz, the first integral is bounded by

$$C\sum_{i=1}^{n}\int_{-\infty}^{+\infty}\left|\check q_i(x) - q_i(x)\right| dx \le C\left(\left\|\check u - u\right\|_{\mathbf{L^1}} + \left\|\check{\tilde u} - \tilde u\right\|_{\mathbf{L^1}}\right).$$

Concerning the second integral, observe that by Lemma 3.8

$$\begin{aligned}
&\mathbf{W}_i[\check u, \check{\tilde u}]\left(q_i(x), x\right) - \mathbf{W}_i[u, \tilde u]\left(q_i(x), x\right) \\
&\qquad \le \kappa_1 C\left\{\left\|\check u(x) - u(x)\right\| + \left\|\check{\tilde u}(x) - \tilde u(x)\right\|\right\}
\end{aligned}$$

and, since $\left|q_i(x)\right|$ is uniformly bounded, the Lemma is proved. □

Proposition 3.10 *The functional $\mathbf{\Phi}$, defined on $\mathcal{D}^*_\delta \times \mathcal{D}^*_\delta$, is lower semi-continuous with respect to the $\mathbf{L^1}$ norm.*

Proof. Fix $u, \tilde u$ in $\mathcal{D}^*_\delta$. Choose two sequences of piecewise constant maps $u_\nu, \tilde u_\nu$ in $\mathcal{D}^*_\delta$ converging to $u, \tilde u$ in $\mathbf{L^1}$. We want to show that $\mathbf{\Phi}(u, \tilde u) \le \liminf_{\nu\to+\infty}\mathbf{\Phi}(u_\nu, \tilde u_\nu)$. Call $l = \liminf_{\nu\to+\infty}\mathbf{\Phi}(u_\nu, \tilde u_\nu)$ and note that, up to subsequences, we may assume that $\lim_{\nu\to+\infty}\mathbf{\Phi}(u_\nu, \tilde u_\nu) = l$. By possibly selecting a further subsequence, we may also assume that both u_ν and $\tilde u_\nu$ converge a.e. to u and $\tilde u$.

Let $\{x_1, \ldots, x_{N+1}\}$ be the set of the jump points in u and $\tilde u$ and write

$$u = \sum_{\alpha=1}^{N} u_\alpha \, \chi_{[x_\alpha, x_{\alpha+1}[}, \qquad \tilde u = \sum_{\alpha=1}^{N} \tilde u_\alpha \, \chi_{[x_\alpha, x_{\alpha+1}[} \, .$$

For all α, select $y_\alpha \in \,]x_\alpha, x_{\alpha+1}[$ so that as $\nu \to +\infty$, the sequence $u_\nu(y_\alpha)$ converges to $u(y_\alpha) = u_\alpha$ and $\tilde u_\nu(y_\alpha)$ converges to $\tilde u(y_\alpha) = \tilde u_\alpha$. Introduce the piecewise constant function $\check u_\nu = \sum_\alpha u_\nu(y_\alpha) \, \chi_{[x_\alpha, x_{\alpha+1}[}$. Let $\check{\tilde u}_\nu$ be defined analogously. Observe that $\check u_\nu$ and $\check{\tilde u}_\nu$ converge pointwise and in $\mathbf{L^1}$ to respectively u and $\tilde u$. Remark 3.5 implies $\lim_{\nu \to +\infty} \mathbf{\Phi} \left(\check u_\nu, \check{\tilde u}_\nu \right) = \mathbf{\Phi} \left(u, \tilde u \right)$, while Lemma 3.9 implies

$$\mathbf{\Phi}(\check u_\nu, \check{\tilde u}_\nu) \le \mathbf{\Phi} \left(u_\nu, \tilde u_\nu \right) + C \left(\left\| \check u_\nu - u_\nu \right\|_{\mathbf{L^1}} + \left\| \check{\tilde u}_\nu - \tilde u_\nu \right\|_{\mathbf{L^1}} \right) .$$

Therefore, passing to the limit $\nu \to +\infty$, the proof is completed:

$$\begin{aligned} \mathbf{\Phi}(u, \tilde u) &= \mathbf{\Phi}(u, \tilde u) - \mathbf{\Phi}(\check u_\nu, \check{\tilde u}_\nu) + \mathbf{\Phi}(\check u_\nu, \check{\tilde u}_\nu) - \mathbf{\Phi}(u_\nu, \tilde u_\nu) + \mathbf{\Phi}(u_\nu, \tilde u_\nu) \\ &\le \mathbf{\Phi}(u, \tilde u) - \mathbf{\Phi}(\check u_\nu, \check{\tilde u}_\nu) + C \left(\left\| \check u_\nu - u_\nu \right\|_{\mathbf{L^1}} + \left\| \check{\tilde u}_\nu - \tilde u_\nu \right\|_{\mathbf{L^1}} \right) \\ &\quad + \mathbf{\Phi}(u_\nu, \tilde u_\nu) \, . \end{aligned}$$

□

Lemma 3.11 *For all $u, \tilde u \in \mathcal{D}_\delta^*$, one has $\mathbf{\Xi}(u, \tilde u) \ge \mathbf{\Phi}(u, \tilde u)$.*

Proof. By the definition (3.4) of $\mathbf{\Xi}$, for all $u, \tilde u \in \mathcal{D}_\delta^*$, there exist sequences $v_\nu, \tilde v_\nu$ of piecewise constant functions such that for $\nu \to +\infty$ we have $v_\nu \to u$, $\tilde v_\nu \to u$ in $\mathbf{L^1}$ and $\mathbf{\Phi}(v_\nu, \tilde v_\nu) \to \mathbf{\Xi}(u, \tilde u)$. Hence, by Proposition 3.10

$$\mathbf{\Phi}(u, \tilde u) \le \liminf_{\nu \to +\infty} \mathbf{\Phi}(v_\nu, \tilde v_\nu) = \mathbf{\Xi}(u, \tilde u) \, ,$$

completing the proof. □

Lemma 3.7 and Lemma 3.11 together yield the following proposition.

Proposition 3.12 *For all u, $\tilde u$ in $\mathcal{D}_\delta^*$, one has $\mathbf{\Xi}(u, \tilde u) = \mathbf{\Phi}(u, \tilde u)$.*

Thanks to the definition (3.4) of $\mathbf{\Xi}$, we obtain

Proposition 3.13 *The functional $\mathbf{\Xi} \colon \mathcal{D}_\delta \times \mathcal{D}_\delta \mapsto \mathbb{R}$ is lower semicontinuous with respect to the $\mathbf{L^1}$ norm.*

Proof. Fix u and $\tilde{u}$ in $\mathcal{D}_\delta$. Let u_ν, respectively $\tilde{u}_\nu$, be a sequence in $\mathcal{D}_\delta$ converging to u, respectively $\tilde{u}$. Define $\varepsilon_\nu = \|u_\nu - u\|_{\mathbf{L^1}} + \|\tilde{u}_\nu - \tilde{u}\|_{\mathbf{L^1}} + 1/\nu$. Then, for each ν, there exist piecewise constant $v_\nu \in B_{\varepsilon_\nu}(u_\nu)$, respectively $\tilde{v}_\nu \in B_{\varepsilon_\nu}(\tilde{u}_\nu)$, such that

$$\mathbf{\Phi}(v_\nu, \tilde{v}_\nu) \leq \mathbf{\Xi}_{\varepsilon_\nu}(u_\nu, \tilde{u}_\nu) + \varepsilon_\nu \leq \mathbf{\Xi}(u_\nu, \tilde{u}_\nu) + \varepsilon_\nu \,. \tag{3.5}$$

Moreover

$$\begin{array}{rcccl} \|v_\nu - u\|_{\mathbf{L^1}} & \leq & \|v_\nu - u_\nu\|_{\mathbf{L^1}} + \|u_\nu - u\|_{\mathbf{L^1}} & < & 2\varepsilon_\nu \\ \|\tilde{v}_\nu - \tilde{u}\|_{\mathbf{L^1}} & \leq & \|\tilde{v}_\nu - \tilde{u}_\nu\|_{\mathbf{L^1}} + \|\tilde{u}_\nu - \tilde{u}\|_{\mathbf{L^1}} & < & 2\varepsilon_\nu \end{array}$$

so that $v_\nu \in B_{2\varepsilon_\nu}(u)$ and $\tilde{v}_\nu \in B_{2\varepsilon_\nu}(\tilde{u})$. Hence, $\mathbf{\Xi}_{2\varepsilon_\nu}(u, \tilde{u}) \leq \mathbf{\Phi}(v_\nu, \tilde{v}_\nu)$. Using (3.5), we obtain $\mathbf{\Xi}_{2\varepsilon_\nu}(u, \tilde{u}) \leq \mathbf{\Xi}(u_\nu, \tilde{u}_\nu) + \varepsilon_\nu$. Finally, passing to the lower limit for $\nu \to +\infty$, we have $\mathbf{\Xi}(u, \tilde{u}) \leq \liminf_{\nu \to +\infty} \mathbf{\Xi}(u_\nu, \tilde{u}_\nu)$. □

In the next proposition, we compare the functional $\mathbf{\Phi}$ defined in (3.3) with the stability functional Φ^ε as defined in [4, formula (8.6)]

Proposition 3.14 *Let $\delta > 0$. Then, there exists a positive C such that for all $\varepsilon > 0$ sufficiently small and for all ε-approximate front tracking solutions $w(t,x), \tilde{w}(t,x)$ of* (1.1)

$$\left| \mathbf{\Phi}\left(w(t,\cdot), \tilde{w}(t,\cdot)\right) - \Phi^\varepsilon(w, \tilde{w})(t) \right| \leq C \cdot \varepsilon \cdot \left\| w(t,\cdot) - \tilde{w}(t,\cdot) \right\|_{\mathbf{L^1}} .$$

Proof. Setting $\tilde{w}(t,x) = \mathbf{S}\left(\mathbf{q}(t,x)\right)\left(w(t,x)\right)$ and omitting the explicit time dependence in the integrand, we have:

$$\begin{aligned} &\left| \mathbf{\Phi}\left(w(t,\cdot), \tilde{w}(t,\cdot)\right) - \Phi^\varepsilon(w, \tilde{w})(t) \right| \\ &\qquad \leq \int_{\mathbb{R}} \sum_{i=1}^{n} \left|q_i(x)\right| \left| \mathbf{W}_i[w, \tilde{w}]\left(q_i(x), x\right) - W_i(x) \right| dx \,. \end{aligned}$$

We are thus lead to estimate

$$\begin{aligned} &\left| \mathbf{W}_i[w, \tilde{w}]\left(q_i(x), x\right) - W_i(x) \right| \\ &\qquad \leq \kappa_1 \left| \mathbf{A}_i[w]\left(q_i(x), x\right) + \mathbf{A}_i[\tilde{w}]\left(-q_i(x), x\right) - A_i(x) \right| \\ &\qquad\qquad + \kappa_1\kappa_2 \left| \mathbf{Q}(w) - Q(w) \right| + \kappa_1\kappa_2 \left| \mathbf{Q}(\tilde{w}) - Q(\tilde{w}) \right| . \end{aligned}$$

The second and third terms in the right hand side are each bounded as in [4, formula (7.100)] by $C\,\varepsilon$. Concerning the former one, recall that, except when $q_i(x) = 0$ or on a finite number of points where w or $\tilde{w}$ have jumps, A_i and $\mathbf{A}_i$ differ only in the absence of non physical waves in A_i. In other words, physical jumps are counted in the same way in both A_i and $\mathbf{A}_i$ while non physical waves appear in $\mathbf{A}_i$ but not in A_i. Therefore, the former term is

almost everywhere bounded, when $q_i(x) \neq 0$, by the sum of the strengths of all non physical waves, i.e. $C\,\varepsilon$ by [4, formula (7.11)]. Finally, using [4, formula (8.5)]:

$$\frac{1}{C} \cdot \left\|w(t,x) - \tilde w(t,x)\right\| \leq \sum_{i=1}^{n} \left|q_i(t,x)\right| \leq C \cdot \left\|w(t,x) - \tilde w(t,x)\right\| \tag{3.6}$$

we complete the proof with the following estimate:

$$\left|\mathbf{\Phi}\left(w(t,\cdot), \tilde w(t,\cdot)\right) - \Phi^\varepsilon(w,\tilde w)(t)\right| \leq C\,\varepsilon \int_{\mathbb{R}} \sum_{i=1}^{n} \left|q_i(x)\right| dx \leq C\,\varepsilon \|w - \tilde w\|_{\mathbf{L^1}}.$$

□

Proof of Theorem 3.6. The estimates [4, formula (8.5)] show that $\mathbf{\Phi}$ is equivalent to the $\mathbf{L^1}$ distance between functions in $\mathcal{D}_\delta^*$. Indeed, if δ is sufficiently small, then $\mathbf{W}_i \in [1,2]$ for all $i = 1,\ldots,n$ and all $x \in \mathbb{R}$, so that

$$\frac{1}{C} \cdot \|v - \tilde v\|_{\mathbf{L^1}} \leq \mathbf{\Phi}(v,\tilde v) \leq 2C \cdot \|v - \tilde v\|_{\mathbf{L^1}}\,. \tag{3.7}$$

To prove *(i)*, fix $u, \tilde u \in \mathcal{D}_\delta$ and choose $v \in B_\eta(u)$, $\tilde v \in B_\eta(\tilde u)$. By (3.7),

$$\begin{array}{ccccc}
\frac{1}{C} \cdot \left(\|u - \tilde u\|_{\mathbf{L^1}} - 2\eta\right) & \leq & \mathbf{\Phi}(v,\tilde v) & \leq & 2C \cdot \left(\|u - \tilde u\|_{\mathbf{L^1}} + 2\eta\right) \\
\frac{1}{C} \cdot \left(\|u - \tilde u\|_{\mathbf{L^1}} - 2\eta\right) & \leq & \mathbf{\Xi}_\eta(u,\tilde u) & \leq & 2C \cdot \left(\|u - \tilde u\|_{\mathbf{L^1}} + 2\eta\right)\,.
\end{array}$$

The proof of *(i)* is completed passing to the limit $\eta \to 0+$.

To prove *(ii)*, fix $u, \tilde u \in \mathcal{D}_\delta$ and $\eta > 0$. Correspondingly, choose $v_\eta \in B_\eta(u)$ and $\tilde v_\eta \in B_\eta(\tilde u)$ satisfying

$$\mathbf{\Xi}(u,\tilde u) \geq \mathbf{\Xi}_\eta(u,\tilde u) \geq \Phi(v_\eta, \tilde v_\eta) - \eta\,. \tag{3.8}$$

Let now $\varepsilon > 0$ and introduce the ε-approximate solutions v_η^ε and $\tilde v_\eta^\varepsilon$ with initial data $v_\eta^\varepsilon(0,\cdot) = v_\eta$ and $\tilde v_\eta^\varepsilon(0,\cdot) = \tilde v_\eta$. Note that for ε sufficiently small

$$\begin{array}{ccccc}
\mathbf{\Upsilon}\left(v_\eta^\varepsilon(t)\right) & \leq & \Upsilon^\varepsilon\left(v_\eta^\varepsilon\right)(t) + C\varepsilon & \leq & \Upsilon^\varepsilon(v_\eta^\varepsilon)(0) + C\varepsilon \\
& \leq & \mathbf{\Upsilon}(v_\eta) + C\varepsilon & < & \delta
\end{array}$$

and an analogous inequality holds for $\tilde v_\eta^\varepsilon$. Therefore $v_\eta^\varepsilon(t)$, $\tilde v_\eta^\varepsilon(t) \in \mathcal{D}_\delta^*$. Here we denoted with Υ^ε the sum $V + C_0 Q$ defined on ε–approximate wave front tracking solutions (see [4, formulæ (7.53), (7.54)]). We may thus apply Proposition 3.12, Proposition 3.14 and the main result in [4, Chapter 8], that is [4, Theorem 8.2], to obtain

$$\begin{array}{rcl}
\mathbf{\Xi}\big(v_\eta^\varepsilon(t), \tilde v_\eta^\varepsilon(t)\big) = & & \mathbf{\Phi}\left(v_\eta^\varepsilon(t), \tilde v_\eta^\varepsilon(t)\right) \\
& \leq & \Phi^\varepsilon\left(v_\eta^\varepsilon, \tilde v_\eta^\varepsilon\right)(t) + C\varepsilon \left\|v_\eta^\varepsilon(t) - \tilde v_\eta^\varepsilon(t)\right\|_{\mathbf{L^1}} \\
& \leq & \Phi^\varepsilon\left(v_\eta^\varepsilon, \tilde v_\eta^\varepsilon\right)(0) + C\varepsilon t + C\varepsilon \left\|v_\eta^\varepsilon(t) - \tilde v_\eta^\varepsilon(t)\right\|_{\mathbf{L^1}} \\
& \leq & \mathbf{\Phi}\left(v_\eta, \tilde v_\eta\right) + C\varepsilon t + C\varepsilon \left\|v_\eta^\varepsilon(t) - \tilde v_\eta^\varepsilon(t)\right\|_{\mathbf{L^1}} + C\varepsilon \left\|v_\eta - \tilde v_\eta\right\|_{\mathbf{L^1}}.
\end{array}$$

Recall that as $\varepsilon \to 0$ by [4, Theorem 8.1] $v_\eta^\varepsilon(t) \to S_t v_\eta$ and $\tilde v_\eta^\varepsilon(t) \to S_t \tilde v_\eta$. Hence, Proposition 3.13 and (3.8) ensure that

$$\mathbf{\Xi}(S_t v_\eta, S_t \tilde v_\eta) \le \liminf_{\varepsilon \to 0+} \mathbf{\Xi}\left(v_\eta^\varepsilon(t), \tilde v_\eta^\varepsilon(t)\right) \le \mathbf{\Phi}\left(v_\eta, \tilde v_\eta\right) \le \mathbf{\Xi}(u, \tilde u) + \eta\,.$$

By the choice of v_η and $\tilde v_\eta$, we have that $v_\eta \to u$ and $\tilde v_\eta \to \tilde u$ in $\mathbf{L^1}$ as $\eta \to 0+$. Therefore, using the continuity of the SRS in $\mathbf{L^1}$ and applying again Proposition 3.13, we may conclude that

$$\mathbf{\Xi}(S_t u, S_t \tilde u) \le \liminf_{\eta \to 0+} \mathbf{\Xi}(S_t v_\eta, S_t \tilde v_\eta) \le \mathbf{\Xi}(u, \tilde u)\,,$$

proving *(ii)*. The latter item *(iii)* follows from Proposition 3.13. □

4 Wave Measures Formulation

Let f satisfy **(F)** and $u \in \mathcal{D}_\delta$ as defined in (2.6). Since $\mathrm{TV}(u)$ is bounded, by possibly changing the values of u at countably many points, we can assume that u is right continuous. Its distributional derivative μ is then a vector measure that can be decomposed into a continuous part μ_c and an atomic one μ_a. For $i = 1, \ldots, n$, consider now the wave measure

$$\mu_i(B) = \int_B l_i(u)\, d\mu_c + \sum_{x \in B} E_i\left(u(x-), u(x+)\right) \tag{4.1}$$

where $l_i(u)$ is the left i-th eigenvector of $Df(u)$, E_i is the i-th component of the map E defined at (2.3) and $B \subseteq \mathbb{R}$ is any Borel set. Here and in what follows, we assume that $l_1, \ldots, l_n$ are normalized so that

$$l_i(u) \cdot r_j(u) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

with r_j as in (2.1). Let μ_i^+, respectively μ_i^- be the positive, respectively negative, part of μ_i and $|\mu_i| = \mu_i^+ + \mu_i^-$ be the total variation of μ_i. Aiming at a definition of the interaction potential by means of the wave measures, introduce the measure

$$\rho = \sum_{1 \le j < i \le n} |\mu_i| \otimes |\mu_j| + \sum_{i=1}^n \left(\mu_i^- \otimes \mu_i^- + \mu_i^+ \otimes \mu_i^- + \mu_i^- \otimes \mu_i^+\right) \tag{4.2}$$

and, as in [3, 4, 5], set

$$\hat{\mathbf{Q}}(u) = \rho\left(\left\{(x,y) \in \mathbb{R}^2 \colon x < y\right\}\right) \tag{4.3}$$

$$\hat{\mathbf{\Upsilon}}(u) = \sum_{i=1}^n |\mu_i|(\mathbb{R}) + C_0 \cdot \hat{\mathbf{Q}}(u)\,. \tag{4.4}$$

For $u, \tilde{u}$ in $\mathcal{D}_\delta$, we now define the functional

$$\hat{\mathbf{\Xi}}(u,\tilde{u}) = \sum_{i=1}^{n} \int_{-\infty}^{+\infty} \left|q_i(x)\right| \hat{\mathbf{W}}_i(x)\, dx \tag{4.5}$$

where the weights $\hat{\mathbf{W}}_i$ are defined by

$$\hat{\mathbf{W}}_i(x) = 1 + \kappa_1 \hat{\mathbf{A}}_i(x) + \kappa_1\kappa_2 \left(\hat{\mathbf{Q}}(u) + \hat{\mathbf{Q}}(\tilde{u})\right) . \tag{4.6}$$

Here, κ_1 and κ_2 are as in [4, Chapter 8], see also (3.2). By means of the wave measures μ_i and $\tilde{\mu}_i$ of u and $\tilde{u}$, if the i-th field is linearly degenerate, define the weights $\hat{\mathbf{A}}_i$ by

$$\begin{aligned} \hat{\mathbf{A}}_i(x) \;\; = & \sum_{1\le i<j\le n} \left|\mu_j\right| \left(\left]-\infty, x\right]\right) + \left|\tilde{\mu}_j\right| \left(\left]-\infty, x\right]\right) \\ & + \sum_{1\le j<i\le n} \left|\mu_j\right| \left(\left]x, +\infty\right[\right) + \left|\tilde{\mu}_j\right| \left(\left]x, +\infty\right[\right) \end{aligned}$$

whereas in the genuinely nonlinear case we let

$$\begin{aligned} \hat{\mathbf{A}}_i(x) \;\; = & \sum_{1\le i<j\le n} \left|\mu_j\right| \left(\left]-\infty, x\right]\right) + \left|\tilde{\mu}_j\right| \left(\left]-\infty, x\right]\right) \\ & + \sum_{1\le j<i\le n} \left|\mu_j\right| \left(\left]x, +\infty\right[\right) + \left|\tilde{\mu}_j\right| \left(\left]x, +\infty\right[\right) \\ & + \left\{ \begin{array}{lll} \displaystyle\sum_{i=1}^{n} \Big(\left|\mu_i\right| \left(\left]-\infty, x\right]\right) + \left|\tilde{\mu}_i\right| \left(\left]x, +\infty\right[\right)\Big) & \text{if } q_i(x) & < \;\; 0 \\ \displaystyle\sum_{i=1}^{n} \Big(\left|\mu_i\right| \left(\left]x, +\infty\right[\right) + \left|\tilde{\mu}_i\right| \left(\left]-\infty, x\right]\right)\Big) & \text{if } q_i(x) & \ge \;\; 0 . \end{array} \right. \end{aligned}$$

On $\mathcal{D}_\delta^*$, $\hat{\mathbf{Q}}$ and $\hat{\mathbf{\Upsilon}}$ obviously coincide respectively with $\mathbf{Q}$ and $\mathbf{\Upsilon}$, therefore also $\mathbf{\Xi}$, $\mathbf{\Phi}$ and $\hat{\mathbf{\Xi}}$ all coincide on $\mathcal{D}_\delta^*$. Below, we prove that $\hat{\mathbf{\Upsilon}} = \mathbf{\Upsilon}$, $\hat{\mathbf{Q}} = \mathbf{Q}$ and $\mathbf{\Xi} = \hat{\mathbf{\Xi}}$ on all $\mathcal{D}_\delta$

The following result is a strengthened version of [4, Lemma 10.1].

Lemma 4.1 *There exists a positive C such that for all $u \in \mathcal{D}_\delta$, $i = 1, \ldots, n$ and $a, b \in \mathbb{R}$ with $a < b$*

$$\left|E_i\left(u(a+), u(b-)\right) - \mu_i\left(\left]a, b\right[\right)\right| \le C \cdot \operatorname{diam}\left(u\left(\left]a, b\right[\right)\right) \cdot |\mu|\left(\left]a, b\right[\right)$$

Proof. We use below the following estimate, see [4, p. 201], valid for all states $u, \tilde{u}$,

$$\left|E_i(u,\tilde{u}) - l_i(u)\cdot(\tilde{u} - u)\right| \le C\left|u - \tilde{u}\right|^2 . \tag{4.7}$$

By the triangle inequality,

$$
\begin{aligned}
&\left|E_i\left(u(a+),u(b-)\right)-\mu_i\left(]a,b[\right)\right| \\
&\qquad \le \left|E_i\left(u(a+),u(b-)\right)-l_i(a+)\cdot\left(u(b-)-u(a+)\right)\right| \\
&\qquad\quad +\left|\mu_i\left(]a,b[\right)-l_i(a+)\cdot\left(u(b-)-u(a+)\right)\right|.
\end{aligned}
$$

The first term in the right hand side is bounded by (4.7). By (4.1), for $I =]a,b[$.

$$
\begin{aligned}
\mu_i(I) \;=\; & \int_I \Big(l_i\left(u(\xi)\right)-l_i\left(u(a+)\right)\Big)\cdot d\mu_c(\xi) \\
&+\sum_{\xi\in I}\Big(E_i\left(u(\xi-),u(\xi+)\right)-l_i\left(u(\xi-)\right)\cdot\left(u(\xi+)-u(\xi-)\right)\Big) \\
&+\sum_{\xi\in I}\Big(l_i\left(u(\xi-)\right)-l_i\left(u(a+)\right)\Big)\cdot\left(u(\xi+)-u(\xi-)\right) \\
&+l_i\left(u(a+)\right)\cdot\left(\sum_{\xi\in I}\left(u(\xi+)-u(\xi-)\right)+\int_I d\mu_c(\xi)\right).
\end{aligned}
$$

We now estimates the different summands above separately. The Lipschitzeanity of l_i ensures that the first summand above is bounded by

$$
\left|\int_I \Big(l_i\left(u(\xi)\right)-l_i\left(u(a+)\right)\Big)\cdot d\mu_c(\xi)\right| \le C\cdot \mathrm{diam}\left(u(I)\right)\cdot|\mu|(I).
$$

Passing to the second summand, using (4.7)

$$
\begin{aligned}
&\left|\sum_{\xi\in I}\Big(E_i\left(u(\xi-),u(\xi+)\right)-l_i\left(u(\xi-)\right)\cdot\left(u(\xi+)-u(\xi-)\right)\Big)\right| \\
\le\;& C\sum_{\xi\in I}\left|u(\xi+)-u(\xi-)\right|^2 \\
\le\;& C\cdot\mathrm{diam}\left(u(I)\right)\cdot|\mu|(I).
\end{aligned}
$$

Using again the Lipschitzeanity of l_i, the third summand is estimated as

$$
\left|\sum_{\xi\in I}\Big(l_i\left(u(\xi-)\right)-l_i\left(u(a+)\right)\Big)\cdot\left(u(\xi+)-u(\xi-)\right)\right| \le C\cdot\mathrm{diam}\left(u(I)\right)\cdot|\mu|(I)
$$

while the last one can be rewritten as

$$
l_i\left(u(a+)\right)\cdot\left(\sum_{\xi\in I}\left(u(\xi+)-u(\xi-)\right)+\int_I d\mu_c(\xi)\right)
$$

$$
\begin{aligned}
&= l_i\left(u(a+)\right)\cdot\left(\mu_a(I)+\mu_c(I)\right)\\
&= l_i\left(u(a+)\right)\cdot\mu(I)\\
&= l_i\left(u(a+)\right)\cdot\left(u(b-)-u(a+)\right)
\end{aligned}
$$

completing the proof. $\square$

Lemma 4.2 *Let $u \in \mathcal{D}_\delta$ with wave measure $\mu_1, \ldots, \mu_n$. Then, there exists a sequence $v_\nu \in \mathcal{D}^*_\delta$ with wave measures $\mu^\nu_1, \ldots, \mu^\nu_n$ such that*

$$
\lim_{\nu\to+\infty}\|v_\nu-u\|_{\mathbf{L}^\infty}=0\,,\qquad \lim_{\nu\to+\infty}\|v_\nu-u\|_{\mathbf{L^1}}=0
$$

$$
\lim_{\nu\to+\infty}\mu_i^{\nu,\pm}(I)=\mu_i^{\pm}(I)\quad \textit{for any interval } I\subseteq\mathbb{R} \tag{4.8}
$$

$$
\lim_{\nu\to+\infty}\hat{\mathbf{Q}}(v_\nu)=\hat{\mathbf{Q}}(u)\,. \tag{4.9}
$$

Moreover, an explicit definition of such a sequence is (4.10)*.*

Proof. For $\nu \in \mathbb{N}\setminus\{0\}$ choose $a, b \in \mathbb{R}$ with $b-a>1$ and so that

$$
\begin{aligned}
|\mu|\left(]-\infty,a]\right)+|\mu|\left([b,+\infty[\right) &\le \frac{1}{\nu}\\
\int_{-\infty}^{a}\left\|u(x)\right\|dx+\int_{b}^{+\infty}\left\|u(x)\right\|dx &\le \frac{1}{\nu}
\end{aligned}
$$

Choose a finite sequence of real numbers $x_1, \ldots, x_N$ so that $a = x_1 < x_2 < \cdots < x_{N-1} < x_N = b$ and $|\mu|\left(]x_{\alpha-1}, x_\alpha[\right) \le \frac{1}{(b-a)\nu}$. Introduce the points $y_0 = x_1 - 1$, $y_\alpha = (x_\alpha + x_{\alpha+1})/2$ for $\alpha = 1, \ldots, N-1$ and $y_N = x_N + 1$. Let

$$
v_\nu=\sum_{\alpha=1}^{N}\left(u(x_\alpha-)\,\chi_{[y_{\alpha-1},x_\alpha[}+u(x_\alpha+)\,\chi_{[x_\alpha,y_\alpha[}\right)\,. \tag{4.10}
$$

Due to the above definitions, the $\mathbf{L^1}$ and $\mathbf{L}^\infty$ convergence $v_\nu \to u$ is immediate (observe that both v_ν and u are right continuous).

We first consider the intervals $I =]-\infty, x]$ for $x \in \mathbb{R}$ or $I = \mathbb{R}$. Let μ_i^ν be the wave measure corresponding to v_ν. For notational simplicity, below we set $x_0 = -\infty$, $x_{N+1} = +\infty$, $u(x_0+) = 0$ and $u(x_{N+1}-) = 0$. Let $\bar\alpha$ be such that $x \in [x_{\bar\alpha}, x_{\bar\alpha+1}[$ ($\bar\alpha = N$ for $I = \mathbb{R}$). For any $i = 1, \ldots, n$,

$$
\begin{aligned}
\left|\mu_i^\nu\right|(I)-|\mu_i|(I) \;=\; &\sum_{\alpha=1}^{\bar\alpha}\left|E_i\left(u(x_\alpha-),u(x_\alpha+)\right)\right|\\
&+\sum_{\alpha=1}^{\bar\alpha}\left|E_i\left(u(x_{\alpha-1}+),u(x_\alpha-)\right)\right|\\
&+\left|E_i\left(u(x_{\bar\alpha}+),u(x_{\bar\alpha+1}-)\right)\right|\chi_{[y_{\bar\alpha},x_{\bar\alpha+1}[}(x)
\end{aligned} \tag{4.11}
$$

$$
\begin{aligned}
&-\sum_{\alpha=1}^{\bar\alpha} |\mu_i| \left(\{x_\alpha\}\right) - \sum_{\alpha=1}^{\bar\alpha} |\mu_i| \left(]x_{\alpha-1}, x_\alpha[\right) \\
&-|\mu_i| \left(]x_{\bar\alpha}, x]\right) \,.
\end{aligned}
$$

Observe that

$$
\begin{array}{rcll}
E_i\left(u(x_\alpha-), u(x_\alpha+)\right) & = & \mu_i\left(\{x_\alpha\}\right) & \text{for all} \quad \alpha = 1,\ldots,N \\
\left|E_i\left(u(x_\alpha-), u(x_\alpha+)\right)\right| & = & |\mu_i|\left(\{x_\alpha\}\right) & \text{for all} \quad \alpha = 1,\ldots,N \\
\left|E_i\left(u(x_{\bar\alpha}+), u(x_{\bar\alpha+1}-)\right)\right| & \le & C\,|\mu|\left(]x_{\bar\alpha}, x_{\bar\alpha+1}[\right) \le C/\nu & \\
|\mu_i|\left(]x_{\bar\alpha}, x]\right) & \le & C\,|\mu|\left(]x_{\bar\alpha}, x_{\bar\alpha+1}[\right) \le C/\nu &
\end{array}
$$

so that, by Lemma 4.1

$$
\begin{aligned}
\left|\mu_i^\nu\right|(I) - |\mu_i|(I) &= \sum_{\alpha=1}^{\bar\alpha}\left(\left|E_i\big(u(x_{\alpha-1}+), u(x_\alpha-)\big)\right| - |\mu_i|\left(]x_{\alpha-1}, x_\alpha[\right)\right) + \frac{C}{\nu} \\
&\le \sum_{\alpha=1}^{\bar\alpha}\left(\left|E_i\big(u(x_{\alpha-1}+), u(x_\alpha-)\big)\right| - \left|\mu_i\left(]x_{\alpha-1}, x_\alpha[\right)\right|\right) + \frac{C}{\nu} \\
&\le \sum_{\alpha=1}^{\bar\alpha}\left|E_i\left(u(x_{\alpha-1}+), u(x_\alpha-)\right) - \mu_i\left(]x_{\alpha-1}, x_\alpha[\right)\right| + \frac{C}{\nu} \\
&\le C\cdot\sum_{\alpha=1}^{\bar\alpha}\operatorname{diam}\Big(u\left(]x_{\alpha-1}, x_\alpha[\right)\Big)\,|\mu|\left(]x_{\alpha-1}, x_\alpha[\right) + \frac{C}{\nu} \\
&\le C\cdot\sum_{\alpha=1}^{\bar\alpha}\frac{1}{\nu}|\mu|\left(]x_{\alpha-1}, x_\alpha[\right) + \frac{C}{\nu} \\
&\le C\cdot\big(1+|\mu|(\mathbb{R})\big)\cdot\frac{1}{\nu} \qquad (4.12)
\end{aligned}
$$

and $\limsup_{\nu\to+\infty}\left|\mu_i^\nu\right|(I) \le |\mu_i|(I)$.

Passing to the other inequality, introduce the functions

$$
w_i(x) = \mu_i\left(]-\infty, x]\right) \qquad \text{and} \qquad w_i^\nu(x) = \mu_i^\nu\left(]-\infty, x]\right) \,.
$$

and repeat the same computations used in (4.11)–(4.12) to obtain

$$
\left|w_i^\nu(x) - w_i(x)\right| = \left|\mu_i^\nu(I) - \mu_i(I)\right| \le C\cdot\big(1+|\mu|(\mathbb{R})\big)\cdot\frac{1}{\nu}
$$

showing that $w_i^\nu \to w_i$ uniformly on $\mathbb{R}$. By the lower semicontinuity of the total variation

$$
|\mu_i|(I) = \mathrm{TV}\left(w_i,]-\infty, x]\right) \le \liminf_{\nu\to+\infty}\mathrm{TV}\left(w_i^\nu,]-\infty, x]\right) = \liminf_{\nu\to+\infty}\left|\mu_i^\nu\right|(I)
$$

showing that $\left|\mu_i^\nu\right|(I) \to |\mu_i|(I)$ as $\nu \to +\infty$. This convergence, together with the uniform convergence above, imply that

$$\begin{aligned}
\lim_{\nu\to+\infty} \left(\mu_i^{\nu,+}(I) + \mu_i^{\nu,-}(I)\right) &= \mu_i^+(I) + \mu_i^-(I) \\
\lim_{\nu\to+\infty} \left(\mu_i^{\nu,+}(I) - \mu_i^{\nu,-}(I)\right) &= \mu_i^+(I) - \mu_i^-(I)
\end{aligned}$$

which together imply (4.8) for $I = \left]-\infty, x\right]$ or $I = \mathbb{R}$. Let now $\tilde u(x)$ and $\tilde v_\nu(x)$ be the right continuous representative of respectively $u(-x)$ and $v_\nu(-x)$ with corresponding wave measures $\tilde\mu_i$ and $\tilde\mu_i^\nu$. The previous computation show that $\tilde\mu_i^{\nu,\pm}\left(\left]-\infty,x\right]\right) \to \tilde\mu_i^{\pm}\left(\left]-\infty,x\right]\right)$ for all real x. But $\tilde\mu_i^{\nu,\pm}\left(\left]-\infty,x\right]\right) = \mu_i^{\nu,\pm}\left(\left[-x,+\infty\right[\right)$ and $\tilde\mu_i^{\pm}\left(\left]-\infty,x\right]\right) = \mu_i^{\pm}\left(\left[-x,+\infty\right[\right)$. Therefore, (4.8) holds also for the intervals $I = \left[x,+\infty\right[$ and therefore for all real intervals $I \subseteq \mathbb{R}$.

Passing to (4.9) we observe that it is enough to show the convergence of every single term in the sum (4.2) which defines the measure ρ. Since the computations for these terms are identical, we show the convergence of only one, say $\mu_i^{\nu,+} \otimes \mu_i^{\nu,-}$. Fix $\varepsilon > 0$ and choose a finite set of real numbers $x_1, \ldots, x_{N_\varepsilon}$ so that $-\infty = x_0^\varepsilon < x_1^\varepsilon < \cdots < x_{N_\varepsilon}^\varepsilon < x_{N_\varepsilon+1}^\varepsilon = +\infty$ and $|\mu|\left(\left]x_{\alpha-1}^\varepsilon, x_\alpha^\varepsilon\right[\right) \leq \varepsilon$. To simplify the notations, we define

$$K = \left\{(x,y) \in \mathbb{R}^2 \colon x < y\right\}, \quad \tau^\nu = \mu_i^{\nu,+} \otimes \mu_i^{\nu,-} \text{ and } \quad \tau = \mu_i^+ \otimes \mu_i^-.$$

Now, write K as the union of a finite family of disjoint sets as:

$$R_{\alpha,\beta} = \left[x_\alpha^\varepsilon, x_{\alpha+1}^\varepsilon\right[\times \left[x_\beta^\varepsilon, x_{\beta+1}^\varepsilon\right[, \quad K = \left(\bigcup_{\substack{\alpha,\beta=0\\ \alpha<\beta}}^{N_\varepsilon} R_{\alpha,\beta}\right) \cup \bigcup_{\alpha=0}^{N_\varepsilon} \left(R_{\alpha,\alpha} \cap K\right)$$

and compute

$$\begin{aligned}
\left|\tau^\nu(K) - \tau(K)\right| \leq & \sum_{\substack{\alpha,\beta=0\\ \alpha<\beta}}^{N_\varepsilon} \left|\tau^\nu(R_{\alpha,\beta}) - \tau(R_{\alpha,\beta})\right| \\
& + \sum_{\alpha=0}^{N_\varepsilon} \left(\tau^\nu(R_{\alpha,\alpha} \cap K) + \tau(R_{\alpha,\alpha} \cap K)\right).
\end{aligned}$$

We have also the estimate

$$\begin{aligned}
\tau^\nu(R_{\alpha,\alpha} \cap K) \leq \tau^\nu\Big(R_{\alpha,\alpha} \setminus \left\{(x_\alpha^\varepsilon, x_\alpha^\varepsilon)\right\}\Big) &= \tau^\nu(R_{\alpha,\alpha}) - \tau^\nu\Big(\left\{(x_\alpha^\varepsilon, x_\alpha^\varepsilon)\right\}\Big) \\
\tau(R_{\alpha,\alpha} \cap K) \leq \tau\left(R_{\alpha,\alpha} \setminus \left\{(x_\alpha^\varepsilon, x_\alpha^\varepsilon)\right\}\right) &= \tau(R_{\alpha,\alpha}) - \tau\left(\left\{(x_\alpha^\varepsilon, x_\alpha^\varepsilon)\right\}\right)
\end{aligned}$$

The limit (4.8) implies that the product measure converges on rectangles:

$$\begin{aligned}
\lim_{\nu\to+\infty} \tau^\nu(R_{\alpha,\beta}) &= \tau\left(R_{\alpha,\beta}\right) \\
\lim_{\nu\to+\infty} \tau^\nu\left(\left\{(x_\alpha^\varepsilon, x_\alpha^\varepsilon)\right\}\right) &= \tau\left(\left\{(x_\alpha, x_\alpha)\right\}\right).
\end{aligned}$$

Therefore

$$\begin{aligned}
\limsup_{\nu\to+\infty}\left|\tau^\nu(K)-\tau(K)\right| &\le 2\sum_{\alpha=0}^{N_\varepsilon}\left(\tau(R_{\alpha,\alpha})-\tau\left(\left\{(x_\alpha^\varepsilon,x_\alpha^\varepsilon)\right\}\right)\right)\\
&= 2\sum_{\alpha=0}^{N_\varepsilon}\Bigg(\mu_i^+\left(\left\{x_\alpha^\varepsilon\right\}\right)\mu_i^-\left(\left]x_\alpha^\varepsilon,x_{\alpha+1}^\varepsilon\right[\right)\\
&\quad +\mu_i^+\left(\left]x_\alpha^\varepsilon,x_{\alpha+1}^\varepsilon\right[\right)\mu_i^-\left(\{x_\alpha^\varepsilon\}\right)\\
&\quad +\mu_i^+\left(\left]x_\alpha^\varepsilon,x_{\alpha+1}^\varepsilon\right[\right)\mu_i^-\left(\left]x_\alpha^\varepsilon,x_{\alpha+1}^\varepsilon\right[\right)\Bigg)\\
&\le C\sum_{\alpha=0}^{N_\varepsilon}\varepsilon|\mu|\left(\left[x_\alpha^\varepsilon,x_{\alpha+1}^\varepsilon\right[\right)\\
&\le C\varepsilon|\mu|(\mathbb{R})\,.
\end{aligned}$$

By the arbitrariness of ε, $\lim_{\nu\to+\infty}\tau^\nu(K)=\tau(K)$, completing the proof. □

The following proof, simpler than that in [4, Theorem 10.1], is based on the piecewise constant approximations introduced in Lemma 4.2.

Proposition 4.3 *The functionals* $\hat{\mathbf{Q}}$ *and* $\hat{\boldsymbol{\Upsilon}}$ *defined in* (4.3) *and* (4.4) *are lower semicontinuous with respect to the* $\mathbf{L}^1$ *norm.*

Proof. We prove the lower semicontinuity only of $\hat{\boldsymbol{\Upsilon}}$ since the other one is easier. Take $u_\nu,\ u\in\mathcal{D}_\delta$ such that $u_\nu\to u$ in $\mathbf{L^1}$. By Lemma 4.2, there exists a sequence of piecewise constant functions $v_\nu\in\mathcal{D}_\delta^*$ such that

$$\|v_\nu-u_\nu\|_{\mathbf{L^1}}\le\frac{1}{\nu}\,,\qquad \hat{\boldsymbol{\Upsilon}}(v_\nu)\le\hat{\boldsymbol{\Upsilon}}(u_\nu)+\frac{1}{\nu}\,.$$

If we define $l=\liminf_{\nu\to+\infty}\hat{\boldsymbol{\Upsilon}}(v_\nu)$, then $l\le\liminf_{\nu\to+\infty}\hat{\boldsymbol{\Upsilon}}(u_\nu)$. Moreover, by possibly passing to a subsequence, we suppose that $l=\lim_{\nu\to+\infty}\hat{\boldsymbol{\Upsilon}}(v_\nu)$ and that $v_\nu(x)\to u(x)$ for every $x\in D$, where $\mathbb{R}\setminus D$ has zero Lebesgue measure. Fix now $\varepsilon>0$ and, using again Lemma 4.2, choose a piecewise constant function $v_\varepsilon\in\mathcal{D}_\delta^*$ which approximates u:

$$v_\varepsilon=\sum_{\alpha=1}^{N_\varepsilon}\left(u(x_\alpha^\varepsilon-)\,\chi_{\left[y_{\alpha-1}^\varepsilon,x_\alpha^\varepsilon\right[}+u(x_\alpha^\varepsilon+)\,\chi_{\left[x_\alpha^\varepsilon,y_\alpha^\varepsilon\right[}\right)$$

$$\hat{\boldsymbol{\Upsilon}}(u)\le\hat{\boldsymbol{\Upsilon}}\left(v_\varepsilon\right)+\varepsilon=\boldsymbol{\Upsilon}\left(v_\varepsilon\right)+\varepsilon\,.$$

By Remark 2.1, for every $\alpha=1,\ldots,N_\varepsilon$, we can choose points

$$\check{x}_\alpha^{\varepsilon-}\in\left]y_{\alpha-1}^\varepsilon,x_\alpha^\varepsilon\right[\cap D\quad\text{ and }\quad\check{x}_\alpha^{\varepsilon+}\in\left]x_\alpha^\varepsilon,y_\alpha^\varepsilon\right[\cap D$$

such that the function

$$\bar{v}_\varepsilon = \sum_{\alpha=1}^{N_\varepsilon} \left(u(\check{x}_\alpha^{\varepsilon-})\, \chi_{\left[y_{\alpha-1}^\varepsilon, x_\alpha^\varepsilon\right[} + u(\check{x}_\alpha^{\varepsilon+})\, \chi_{\left[x_\alpha^\varepsilon, y_\alpha^\varepsilon\right[} \right)$$

satisfies $\mathbf{\Upsilon}(v_\varepsilon) \leq \mathbf{\Upsilon}\left(\bar{v}_\varepsilon\right) + \varepsilon$. Define now

$$\check{v}_{\varepsilon,\nu} = \sum_{\alpha=1}^{N_\varepsilon} \left(v_\nu(\check{x}_\alpha^{\varepsilon-})\, \chi_{\left[y_{\alpha-1}^\varepsilon, x_\alpha^\varepsilon\right[} + v_\nu(\check{x}_\alpha^{\varepsilon+})\, \chi_{\left[x_\alpha^\varepsilon, y_\alpha^\varepsilon\right[} \right).$$

Since $v_\nu\left(\check{x}_\alpha^{\varepsilon\pm}\right) \to u\left(\check{x}_\alpha^{\varepsilon\pm}\right)$, we can apply again Remark 2.1 to obtain that for ν sufficiently large one has $\mathbf{\Upsilon}\left(\bar{v}_\varepsilon\right) \leq \mathbf{\Upsilon}\left(\check{v}_{\varepsilon,\nu}\right) + \varepsilon$.

But $\check{v}_{\varepsilon,\nu}$ is obtained by removing an ordered sequence of values attained by v_ν, therefore we can apply Proposition 3.2 to get $\mathbf{\Upsilon}\left(\check{v}_{\varepsilon,\nu}\right) \leq \mathbf{\Upsilon}\left(v_\nu\right)$ and therefore we have the following chain of inequalities:

$$\hat{\mathbf{\Upsilon}}\left(u\right) \leq \mathbf{\Upsilon}\left(v_\varepsilon\right) + \varepsilon \leq \mathbf{\Upsilon}\left(\bar{v}_\varepsilon\right) + 2\varepsilon \leq \mathbf{\Upsilon}\left(\check{v}_{\varepsilon,\nu}\right) + 3\varepsilon \leq \mathbf{\Upsilon}\left(v_\nu\right) + 3\varepsilon\,.$$

Taking the limit as $\nu \to +\infty$ one obtains $\hat{\mathbf{\Upsilon}}\left(u\right) \leq l + 3\varepsilon$ and the arbitrariness of $\varepsilon > 0$ implies $\hat{\mathbf{\Upsilon}}\left(u\right) \leq l \leq \liminf_{\nu\to+\infty} \hat{\mathbf{\Upsilon}}(u_\nu)$. □

Corollary 4.4 *The functionals $\hat{\mathbf{Q}}$ and $\hat{\mathbf{\Upsilon}}$ defined by* (4.3) *and* (4.4) *coincide on all $\mathcal{D}_\delta$ with $\mathbf{Q}$ and $\mathbf{\Upsilon}$.*

Proof. It is a straightforward consequence of the lower semicontinuity of both $\hat{\mathbf{Q}}$ and $\hat{\mathbf{\Upsilon}}$ (Proposition 4.3) and $\mathbf{Q}$ and $\mathbf{\Upsilon}$ (Proposition 3.1).

Indeed, consider only $\hat{\mathbf{\Upsilon}}$ and $\mathbf{\Upsilon}$. They obviously coincide on $\mathcal{D}_\delta^*$. Fix now $u \in \mathcal{D}_\delta$. By the definition (3.1) of $\mathbf{\Upsilon}$, there exists a sequence v_ν of functions in $\mathcal{D}_\delta^*$ converging to u in $\mathbf{L^1}$ and such that $\mathbf{\Upsilon}(v_\nu) \to \mathbf{\Upsilon}(u)$ as $\nu \to +\infty$. By the lower semicontinuity of $\hat{\mathbf{\Upsilon}}$ (Proposition 4.3), we obtain

$$\hat{\mathbf{\Upsilon}}(u) \leq \liminf_{\nu\to+\infty} \hat{\mathbf{\Upsilon}}(v_\nu) = \liminf_{\nu\to+\infty} \mathbf{\Upsilon}(v_\nu) = \lim_{\nu\to+\infty} \mathbf{\Upsilon}(v_\nu) = \mathbf{\Upsilon}(u)\,.$$

Analogously, by Lemma 4.2 we can take a sequence v_ν of functions in $\mathcal{D}_\delta^*$ such that $v_\nu \to u$ in $\mathbf{L^1}$ and $\hat{\mathbf{\Upsilon}}(v_\nu) \to \hat{\mathbf{\Upsilon}}(u)$ as $\nu \to +\infty$. Therefore, along this particular sequence, we may repeat the estimates as above applying the lower semicontinuity of $\mathbf{\Upsilon}$ (Proposition 3.1):

$$\mathbf{\Upsilon}(u) \leq \liminf_{\nu\to+\infty} \mathbf{\Upsilon}(v_\nu) = \liminf_{\nu\to+\infty} \hat{\mathbf{\Upsilon}}(v_\nu) = \lim_{\nu\to+\infty} \hat{\mathbf{\Upsilon}}(v_\nu) = \hat{\mathbf{\Upsilon}}(u).$$

□

By Corollary 4.4, in the following we write $\mathbf{Q}$ and $\mathbf{\Upsilon}$ for $\hat{\mathbf{Q}}$ and $\hat{\mathbf{\Upsilon}}$.

The following proposition shows the lower semicontinuity of $\hat{\mathbf{\Xi}}$ along piecewise constant converging sequences.

Lemma 4.5 *Let $u,\ \tilde{u} \in \mathcal{D}_\delta$. Then, the approximating sequences $v_\nu,\ \tilde{v}_\nu \in \mathcal{D}_\delta^*$ defined in Lemma 4.2 satisfy also* $\lim_{\nu\to+\infty} \hat{\mathbf{\Xi}}(v_\nu, \tilde{v}_\nu) = \hat{\mathbf{\Xi}}(u, \tilde{u})$.

Proof. Define $\mathbf{q}(x)$ and $\mathbf{q}^\nu(x)$ so that $\tilde{u}(x) = \mathbf{S}\left(\mathbf{q}(x)\right)\left(u(x)\right)$ and $\tilde{v}_\nu(x) = \mathbf{S}\left(\mathbf{q}^\nu(x)\right)\left(v_\nu(x)\right)$. Then, $\mathbf{q}^\nu \to \mathbf{q}$ uniformly and in $\mathbf{L^1}$. Let $\mathbf{W}_i(x)$, respectively $\mathbf{W}_i^\nu(x)$, be the weights defined in (4.6) with reference to $u, \tilde{u}$, respectively $v_\nu, \tilde{v}_\nu$. Compute

$$\begin{aligned}\left|\hat{\mathbf{\Xi}}\left(v_\nu, \tilde{v}_\nu\right) - \hat{\mathbf{\Xi}}\left(u, \tilde{u}\right)\right| \quad \leq \quad & \sum_{i=1}^{n} \int_{-\infty}^{+\infty} \left|q_i^\nu(x) - q_i(x)\right| \cdot \mathbf{W}_i^\nu(x)\, dx \\ & + \sum_{i=1}^{n} \int_{-\infty}^{+\infty} \left|q_i(x)\right| \cdot \left|\mathbf{W}_i^\nu(x) - \mathbf{W}_i(x)\right| dx\end{aligned}$$

The first integral converges obviously to zero. Concerning the second one, where $q_i(x) = 0$ the integrand vanishes. Otherwise, if $q_i(x) \neq 0$, then for ν sufficiently large $q_i(x) \cdot q_i^\nu(x) > 0$ and hence the weights depend continuously only on the wave measures and on the interaction potentials which all converge by Lemma 4.2. Therefore, for all $x \in \mathbb{R}$ the integrand satisfies

$$\lim_{\nu\to+\infty} \left|q_i(x)\right| \cdot \left|\mathbf{W}_i^\nu(x) - \mathbf{W}_i(x)\right| = 0\,.$$

The Dominated Convergence Theorem concludes the proof. □

Proposition 4.6 *For any $u,\ \tilde{u} \in \mathcal{D}_\delta$ and $v_\nu,\ \tilde{v}_\nu \in \mathcal{D}_\delta^*$ such that $v_\nu \to u$ and $\tilde{v}_\nu \to \tilde{u}$ in $\mathbf{L^1}$ as $\nu \to +\infty$, we have* $\hat{\mathbf{\Xi}}(u, \tilde{u}) \leq \liminf_{\nu\to+\infty} \hat{\mathbf{\Xi}}(v_\nu, \tilde{v}_\nu)$.

Proof. Let $l = \liminf_{\nu\to+\infty} \hat{\mathbf{\Xi}}(v_\nu, \tilde{v}_\nu)$. Passing to subsequences, we assume that $l = \lim_{\nu\to+\infty} \hat{\mathbf{\Xi}}\left(v_\nu, \tilde{v}_\nu\right)$ and that v_ν, respectively $\tilde{v}_\nu$, converges pointwise to u, respectively $\tilde{u}$, on a set $D \subseteq \mathbb{R}$ with $\mathbb{R}\backslash D$ having zero Lebesgue measure. Fix $\varepsilon > 0$ and apply lemmas 4.2 and 4.5 to find two functions

$$v_\varepsilon = \sum_{\alpha=1}^{N_\varepsilon} \left(u(x_\alpha^\varepsilon -)\, \chi_{\left[y_{\alpha-1}^\varepsilon, x_\alpha^\varepsilon\right[} + u(x_\alpha^\varepsilon +)\, \chi_{\left[x_\alpha^\varepsilon, y_\alpha^\varepsilon\right[} \right)\,,$$

$$\tilde{v}_\varepsilon = \sum_{\alpha=1}^{\tilde{N}_\varepsilon} \left(\tilde{u}(\tilde{x}_\alpha^\varepsilon -)\, \chi_{\left[\tilde{y}_{\alpha-1}^\varepsilon, \tilde{x}_\alpha^\varepsilon\right[} + \tilde{u}(\tilde{x}_\alpha^\varepsilon +)\, \chi_{\left[\tilde{x}_\alpha^\varepsilon, \tilde{y}_\alpha^\varepsilon\right[} \right)\,,$$

such that $\left|\hat{\boldsymbol{\Xi}}(u,\tilde u)-\hat{\boldsymbol{\Xi}}(v_\varepsilon,\tilde v_\varepsilon)\right|+\|u-v_\varepsilon\|_{\mathbf{L^1}}+\|\tilde u-\tilde v_\varepsilon\|_{\mathbf{L^1}}\leq\varepsilon$. On $\mathcal{D}^*_\delta$, $\hat{\boldsymbol{\Xi}}$ and $\boldsymbol{\Phi}$ coincide, hence Remark 3.5 applies and there are points

$$\begin{array}{lcl} \check x_\alpha^{\varepsilon-}\in\,]y^\varepsilon_{\alpha-1},x^\varepsilon_\alpha[\,\cap D & \text{and} & \check x_\alpha^{\varepsilon+}\in\,]x^\varepsilon_{\alpha},y^\varepsilon_\alpha[\,\cap D \\ \check{\tilde x}_\alpha^{\varepsilon-}\in\,]\tilde y^\varepsilon_{\alpha-1},\tilde x^\varepsilon_\alpha[\,\cap D & \text{and} & \check{\tilde x}_\alpha^{\varepsilon+}\in\,]\tilde x^\varepsilon_{\alpha},\tilde y^\varepsilon_\alpha[\,\cap D \end{array}$$

such that the two functions

$$\begin{aligned} \check v_{\varepsilon,\nu} &= \sum_{\alpha=1}^{N_\varepsilon}\left(v_\nu(\check x_\alpha^{\varepsilon-})\,\chi_{[y^\varepsilon_{\alpha-1},x^\varepsilon_\alpha[}+v_\nu(\check x_\alpha^{\varepsilon+})\,\chi_{[x^\varepsilon_\alpha,y^\varepsilon_\alpha[}\right)\\ \check{\tilde v}_{\varepsilon,\nu} &= \sum_{\alpha=1}^{\tilde N_\varepsilon}\left(\tilde v_\nu(\check{\tilde x}_\alpha^{\varepsilon-})\,\chi_{[\tilde y^\varepsilon_{\alpha-1},\tilde x^\varepsilon_\alpha[}+\tilde v_\nu(\check{\tilde x}_\alpha^{\varepsilon+})\,\chi_{[\tilde x^\varepsilon_\alpha,\tilde y^\varepsilon_\alpha[}\right) \end{aligned}$$

satisfy $\left|\hat{\boldsymbol{\Xi}}(v_\varepsilon,\tilde v_\varepsilon)-\hat{\boldsymbol{\Xi}}(\check v_{\varepsilon,\nu},\check{\tilde v}_{\varepsilon,\nu})\right|+\left\|\check v_{\varepsilon,\nu}-v_\varepsilon\right\|_{\mathbf{L^1}}+\left\|\check{\tilde v}_{\varepsilon,\nu}-\tilde v_\varepsilon\right\|_{\mathbf{L^1}}\leq\varepsilon$ for ν sufficiently large. Lemma 3.9 thus implies

$$\hat{\boldsymbol{\Xi}}(\check v_{\varepsilon,\nu},\check{\tilde v}_{\varepsilon,\nu})\leq\hat{\boldsymbol{\Xi}}(v_\nu,\tilde v_\nu)+C\left(\left\|\check v_{\varepsilon,\nu}-v_\nu\right\|_{\mathbf{L^1}}+\left\|\check{\tilde v}_{\varepsilon,\nu}-\tilde v_\nu\right\|_{\mathbf{L^1}}\right).$$

Hence

$$\begin{aligned} \hat{\boldsymbol{\Xi}}(u,\tilde u) &\leq \hat{\boldsymbol{\Xi}}(v_\varepsilon,\tilde v_\varepsilon)+\varepsilon\\ &\leq \hat{\boldsymbol{\Xi}}(\check v_{\varepsilon,\nu},\check{\tilde v}_{\varepsilon,\nu})+2\varepsilon\\ &\leq \hat{\boldsymbol{\Xi}}(v_\nu,\tilde v_\nu)+2\varepsilon+C\left(\left\|\check v_{\varepsilon,\nu}-v_\nu\right\|_{\mathbf{L^1}}+\left\|\check{\tilde v}_{\varepsilon,\nu}-\tilde v_\nu\right\|_{\mathbf{L^1}}\right)\\ &\leq \hat{\boldsymbol{\Xi}}(v_\nu,\tilde v_\nu)+2\varepsilon\cdot(C+1)+C\left(\|v_\varepsilon-v_\nu\|_{\mathbf{L^1}}+\|\tilde v_\varepsilon-\tilde v_\nu\|_{\mathbf{L^1}}\right) \end{aligned}$$

so that, for $\nu\to+\infty$,

$$\hat{\boldsymbol{\Xi}}(u,\tilde u)\leq l+2\varepsilon(C+1)+C\left(\|v_\varepsilon-u\|_{\mathbf{L^1}}+\|\tilde v_\varepsilon-\tilde u\|_{\mathbf{L^1}}\right)\leq l+2\varepsilon(2C+1)\,.$$

The arbitrariness of ε concludes the proof. □

Theorem 4.7 *Let f satisfy **(F)**. The functional $\hat{\boldsymbol{\Xi}}$ defined in* (4.5) *coincides on all $\mathcal{D}_\delta$ with $\boldsymbol{\Xi}$ as defined in* (3.4)*. In particular $\hat{\boldsymbol{\Xi}}$ is lower semicontinuous.*

Proof. Both functionals can be approximated through their evaluation on piecewise constant functions (see (3.4) and Lemma 4.5). Both functionals coincide on piecewise constant functions and are lower semicontinuous along sequences of piecewise constant functions (see Theorem 3.6 and Proposition 4.6). A procedure identical to that of Corollary 4.4 completes the proof. □

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